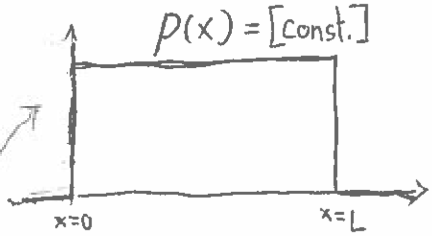
Practice on Probability Distributions (Solutions)

1)



We'll call this constant "c"

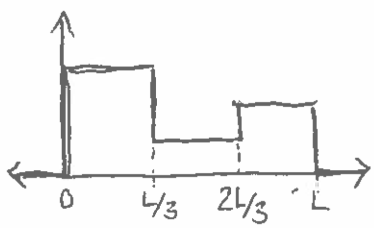
Normalization: $\int_0^L p(x)\,dx = 1 = \int_0^L c\,dx = cL \rightarrow c = \frac{1}{L}$, so $\boxed{p(x) = \frac{1}{L}}$

Mean: $\langle x\rangle = \int_0^L p(x)\,x\,dx = \frac{1}{L}\int_0^L x\,dx = \frac{1}{2L}x^2\Big|_0^L = \boxed{\frac{L}{2}}$

$$\int_0^L p(x)x^2\,dx = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{3L}x^3\Big|_0^L = \boxed{\frac{L^2}{3}}$$

$$\longrightarrow \sigma = \sqrt{\frac{L^2}{3} - \left(\frac{L}{2}\right)^2} = L\sqrt{\frac{1}{3} - \frac{1}{4}} = \boxed{\frac{L}{2\sqrt{3}}} \approx \boxed{0.289L}$$

2)



Normalization: $1 = c\left(\frac{3aL}{3} + \frac{aL}{3} + \frac{2aL}{3}\right) = \frac{6aL}{3} = 2aLc$

$$\longrightarrow c = \frac{1}{2aL}$$

Mean: $\langle x\rangle = \int_0^L p(x)\,x\,dx = c\left[\int_0^{L/3} 3ax\,dx + \int_{L/3}^{2L/3} ax\,dx + \int_{2L/3}^{L} 2ax\,dx\right]$

$$= c\left[\frac{3x^2a}{2}\Big|_0^{L/3} + \frac{x^2a}{2}\Big|_{L/3}^{2L/3} + \frac{2x^2a}{2}\Big|_{2L/3}^{L}\right]$$

$$= \frac{ca}{2}\left[3\left(\frac{L}{3}\right)^2 + \left(\frac{4L^2}{9} - \frac{L^2}{9}\right) + 2\left(L^2 - \frac{4L^2}{9}\right)\right]$$

$$= \frac{ca}{18}\left[3 + 3 + 2\cdot 5\right]L^2 = \left(\frac{1}{2aL}\right)\frac{aL^2}{18}\cdot 16 = \boxed{\frac{4L}{9}}$$

$\langle x^2\rangle$: $\langle x^2\rangle = c\left[\int_0^{L/3} 3ax^2\,dx + \int_{L/3}^{2L/3} ax^2\,dx + \int_{2L/3}^{L} 2ax^2\,dx\right]$

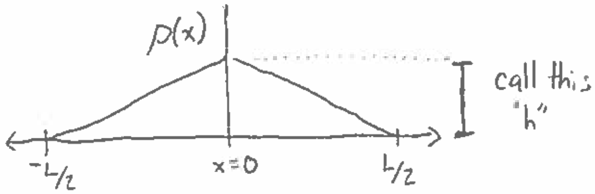
$$= c\left[3a\frac{x^3}{3}\Big|_0^{L/3} + a\frac{x^3}{3}\Big|_{L/3}^{2L/3} + 2a\frac{x^3}{3}\Big|_{2L/3}^{L}\right]$$

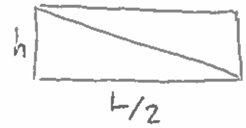
$$= \frac{ca}{3}\left[3\left(\frac{L}{3}\right)^3 + \left(\frac{2L}{3}\right)^3 - \left(\frac{L}{3}\right)^3 + 2L^3 - 2\left(\frac{2L}{3}\right)^3\right]$$

$$= \frac{a}{3\cdot 2aL}\left(\frac{L}{3}\right)^3\left[3 + 2^3 - 1 + 2\cdot 3^3 - 2\cdot 2^3\right] = \frac{L^2}{162}\left[48\right] = \boxed{\frac{8L^2}{27}}$$

$$\rightarrow \sigma = \sqrt{\frac{8L^2}{27}\ \left(\frac{4L}{9}\right)^2} = L\sqrt{\frac{8}{81}} = \boxed{\frac{2\sqrt{2}}{9}L} \approx \boxed{0.314L}$$

3)



Normalization: 

} Total area: $h \cdot \frac{L}{2} =$

so it must be that $h = \frac{2}{L}$

• Slope is therefore $\frac{\Delta y}{\Delta x} = \frac{-h}{(L/2)} = \frac{-2}{L}\left(\frac{2}{L}\right) = \frac{-4}{L^2}$

• So $\boxed{p(x) = \frac{-4}{L^2}x + \frac{2}{L}}$ for $x>0$ and $\boxed{p(x) = \frac{4}{L^2}x + \frac{2}{L}}$ for $x<0$

or, together, $\boxed{p(x) = \frac{-4}{L^2}|x| + \frac{2}{L}}$

Mean: $\langle x\rangle = \int_{-L/2}^{L/2}\left(-\frac{4}{L^2}|x| + \frac{2}{L}\right)x\,dx = \int_{-L/2}^{0}\left(\frac{4}{L^2}x^2 + \frac{2x}{L}\right)dx + \int_{0}^{L/2}\left(\frac{4}{L^2}x^2 + \frac{2x}{L}\right)dx$

$$= \frac{-4}{3L^2}\left(\frac{-L}{2}\right)^3 + \left(\frac{-4}{3L^2}\right)\left(\frac{L}{2}\right)^3 - \frac{(-L/2)^2}{L} + \frac{1}{L}\left(\frac{L}{2}\right)^2 = \boxed{0}$$

(as we'd expect from the symmetry of the problem)

$\langle x^2\rangle$: $\int_{-L/2}^{0}\left(\frac{4x^3}{L^2} + \frac{2x^2}{L}\right)dx + \int_{0}^{L/2}\left(-\frac{4x^3}{L^2} + \frac{2x^2}{L}\right)dx$

$$= -\frac{(-L/2)^4}{L^2} - \frac{2}{3L}\left(\frac{-L}{2}\right)^3 - \frac{(L/2)^4}{L^2} + \frac{2(L/2)^3}{3L} = -\frac{L^2}{8} + \frac{4}{3}\frac{L^2}{8} = \frac{L^2}{24}$$

$\longrightarrow \sigma = \sqrt{\frac{L^2}{24} - 0} = \boxed{\frac{L}{2\sqrt{6}}} \approx \boxed{0.204\,L}$