# Appendix A

# Mean and Standard Deviation

## A.1 INTRODUCTION

Suppose you carry out a series of $N$ measurements of a quantity $x$. The individual measurements are labeled $x_i$, where $i$ is an integer from 1 to $N$ labeling the individual measurements. Ideally, all the measurements would be identical. Practically, however, there is always some random scatter in the data. The challenge is then to make the best estimate of the "true" value of $x$ and the uncertainty in that estimate.

## A.2 EXAMPLE

| **Trial** | **Measurement** |
|---|---|
| 1 | 2.45 |
| 2 | 2.44 |
| 3 | 2.46 |
| 4 | 2.48 |
| 5 | 2.47 |
| 6 | 2.45 |
| 7 | 2.46 |
| 8 | 2.44 |
| 9 | 2.43 |
| 10 | 2.45 |

## A.3 MEAN, $\bar{x}$

The simple average of the results of repeated attempts to measure the same quantity is called the experimental *mean*. It is given mathematically by

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \ ,$$

where $N$ is the total number of measurements (10, in this example) and the $x_i$ are the individual measurements.

Most scientific calculators have a special key to calculate the mean for you; often it is indicated by $\bar{x}$.

In `Excel`, you can use the `AVERAGE()` function to calculate the mean. Suppose you have your $x_i$ measurements in cells `B2` through `B11`. To put the average in cell `B13`, go to cell B13 and type

```
=Average(B2:B11).
```

For the data given above, the mean value is $\bar{x} = 2.4530$.

## A.4 STANDARD DEVIATION, $\sigma$

Now that you have found the mean value, it is also important to characterize how much your measured values scatter around that mean. The most commonly used measure of randomness is the *standard deviation*, given mathematically by

$$\sigma = \left[\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2\right]^{1/2} \ .$$

For the data given above, the standard deviation is $\sigma = 0.0149$. (Sometimes this is written $\sigma_{n-1}$ instead of $\sigma$.)

Most scientific calculators have a key for this also; some have two keys, one labeled $\sigma_n$ and another labeled $\sigma_{n-1}$. Make sure you use the latter: a key labeled $\sigma_n$ most likely calculates a similar quantity, but with just $N$ in the denominator, not $N-1$. Consult your calculator manual if you have any doubt.

Some calculators present you with two values: one labeled $S_x$ and one labeled with $\sigma_x$. You want the larger of the two, which is $S_x$ on the calculators I have seen. Check for yourself to be sure—try the data in the table above and see which one gives you the correct answer.

The standard deviation tells us, in effect, how far from the average the next single measurement is likely to be. More specifically, if the data follow a "normal" or "Gaussian" distribution, then we would expect 68% of our measurements to give results in the range $(\bar{x} - \sigma)$ to $(\bar{x} + \sigma)$; 95% should fall within $2\sigma$ of the mean, and 99.7% should fall within $3\sigma$. These limits are usually called "one-sigma," "two-sigma" and "three-sigma."

In `Excel`, you can use the `STDEV()` function to calculate the standard deviation.

## A.5 UNCERTAINTY, $\sigma_{\bar{x}}$

Lastly, we would like to know how reliable our mean value is. If we took another set of ten measurements and averaged those results, how different might the new average be? In other words, what is the *uncertainty of the mean*? A good measure of this, sometimes also called the standard deviation of the mean, is given by

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} \; .$$

The distinction between the *standard deviation*, $\sigma$, and the *uncertainty of the mean*, $\sigma_{\bar{x}}$, is an important one. If we ask how far from the mean the next single measurement is likely to be, it should not matter whether we have done ten measurements before or a hundred measurements; the *standard deviation* should not change (much) when we increase the number of measurements substantially.

On the other hand, if we make very few measurements we would not place too much faith in the average result, but if we average a great many measurements we expect the result to be quite precise. So the *uncertainty* should get smaller and smaller as we increase the number of measurements that go into the mean. The $1/\sqrt{N}$ factor in $\sigma_{\bar{x}}$ does just that.

For the example given above, $\sigma_{\bar{x}}$ is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{10}} = \frac{0.0149}{\sqrt{10}} = 0.0047 \; .$$

In `Excel` you can use the `SQRT()` function to take square roots.

## A.6 REPORTING RESULTS

When you report the results of an experiment, it is very important that you also report a measure of their reliability. Therefore we always report values *with their uncertainties*: $\bar{x} \pm \sigma_{\bar{x}}$, not $\bar{x}$ alone.

For the example given above, you would report your result as

$$2.453 \pm 0.005$$

Notice that the uncertainty is rounded to only one significant digit, and that the mean itself is rounded to the same number of decimal points. It makes no sense to retain any digits beyond these; they are meaningless.

## A.7 ERRANT POINTS

Sometimes one data point in a set will be totally out of line with all the others. Perhaps something went wrong with the equipment; perhaps you misrecorded that data point; perhaps something was significantly different about the system for that one point. A general rule of thumb is that if a data point falls more than $3\sigma$ from the mean (when the mean and standard deviation are calculated *including* that point), you should assume the deviant point is spurious and discard it.

## A.8 COMPARISON WITH EXPECTED VALUES

It is unlikely that your estimate of $\bar{x}$ is exactly the same as the expected value of $x_{theory}$. Suppose, for example, that the theoretical value of $x$ in the above example was $x_{theory} = 2.450$. In order to determine if that difference is significant or not, you need to compare the mean value $\bar{x}$ with the expected value $x_{theory}$. If the difference is less than the measurement uncertainty $\sigma_{\bar{x}}$, then we say the results agree. If the measured and expected values differ by more than $3\sigma_{\bar{x}}$, then we say the results disagree. If the difference is more than $\sigma_{\bar{x}}$ but less than $3\sigma_{\bar{x}}$, then the situation is unclear. Ideally, you would go back and take more data, since random errors will tend to cancel out. Practically, however, you often won't have time to go back and take more data, so you should just state your results clearly and then proceed on.

In the example above, the difference is

$$\bar{x} - x_{theory} = 2.4530 - 2.450 = 0.0030\ .$$

This difference (0.0030) is less than $\sigma_{\bar{x}}$ (0.0047), so we would say these results agree, to within the uncertainties.