Appendix B

Propagation of Uncertainties

The process of determining the uncertainty in a final result derived from a number of component quantities, each of which has its own measurement uncertainty, is called *propagation of uncertainties* or *propagation of errors*. In general this is a complex subject, but for common mathematical relationships among the quantities it can be reduced to a few straightforward rules, all of which have a rigorous mathematical underpinning. You can find the details of the derivation of these rules in Bevington, *Data Reduction and Error Analysis for the Physical Sciences* or any other standard text on the subject. Here we will merely state the rules without derivation. These rules apply only if the component quantities are measured independently.

The notation used is as follows: if A is a measured or derived quantity, then σ_A is the uncertainty (or standard deviation of the mean) in that result. Constants are written as lower case letters (b, c, etc.); measured quantities are written in upper case (A, B, R, etc.).

Some examples are given below.

B.1 RULES

Addition or subtraction: if $R = A + B$, then $\sigma_R = \sqrt{\sigma_A^2 + \sigma_B^2}$

Multiplication by a constant: if $R = cA$, then $\sigma_R = c\sigma_A$, or

$$\frac{\sigma_R}{|R|} = \frac{\sigma_A}{|A|} \; .$$

Division by a constant: if $R = A/b$, then $\sigma_R = \sigma_A/b$, or

$$\frac{\sigma_R}{|R|} = \frac{\sigma_A}{|A|} \; .$$

Reciprocal: if $R = 1/A$, then

$$\frac{\sigma_R}{|R|} = \frac{\sigma_A}{|A|} \; .$$

Multiplication of two quantities: if $R = AB$, then

$$\frac{\sigma_R}{R} = \sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2} \; .$$

Division by another measured quantity: if $R = A/B$, then

$$\frac{\sigma_R}{R} = \sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2} \; .$$

Powers: if $R = cA^b$, then

$$\frac{\sigma_R}{R} = |b| \left(\frac{\sigma_A}{A}\right) \; .$$

Exponentials: if $R = ce^{bA}$, then

$$\frac{\sigma_R}{|R|} = b\sigma_A \; .$$

If $R = c^{bA}$, then

$$\frac{\sigma_R}{|R|} = (b \ln c)\sigma_A \; .$$

Logarithms: if $R = c\ln(bA)$, then

$$\sigma_R = |c| \left(\frac{\sigma_A}{A}\right) \; .$$

Expressions involving multiple operations can usually be broken down into smaller pieces, e.g.: if

$$R = \frac{A+B}{C+D} \; ,$$

then let $R_{num} = A + B$ and $R_{denom} = C + D$. The uncertainties in the numerator and denominator are then

$$\begin{aligned} \sigma_{R_{num}} &= \sqrt{\sigma_A^2 + \sigma_B^2} \\ \sigma_{R_{denom}} &= \sqrt{\sigma_C^2 + \sigma_D^2} \end{aligned}$$

so that

$$R = \frac{R_{num}}{R_{denom}} \quad \text{and}$$

$$\frac{\sigma_R}{R} = \sqrt{\left(\frac{\sigma_{R_{num}}}{R_{num}}\right)^2 + \left(\frac{\sigma_{R_{denom}}}{R_{denom}}\right)^2} \; .$$

B.2 EXAMPLES

B.2.1 Addition and Subtraction

Suppose you have measurements (with uncertainty) of two distances $x = 3.0 \pm 0.4\,\mathrm{m}$ and $y = 2.5 \pm 0.3\,\mathrm{m}$, and you want to find the total distance $d = x + y$. How do you estimate the uncertainty in d?

If you just added the uncertainties together, you would overestimate the uncertainty in the final result: One of the values might be too high while the other might be too low, so the uncertainties might partially cancel. This possibility is taken into account (in a rigorous way) in the rules given above. Thus we will use the rule given above to combine these uncertainties:

$$d = x + y = 3.0 + 2.5 = 5.5$$
$$\sigma_d = \sqrt{\sigma_x^2 + \sigma_y^2} = \sqrt{(0.4)^2 + (0.3)^2} = 0.5$$

Thus you would report the result as $d = 5.5 \pm 0.5\,\mathrm{m}$.

The rule for uncertainties in subtraction is the same.

B.2.2 Multiplication and Division

When multiplying or dividing, you usually use *fractional* or *percent* uncertainties, rather than absolute uncertainties, as was the case for addition and subtraction. Otherwise, the rules are similar.

For example, suppose you have measured the time t it takes a car to travel a distance x on the air track and you want to calculate the velocity v of the car according to the equation

$$v = \frac{x}{t}$$

Further, assume that x is known precisely, but there is some uncertainty σ_t in the time t. How do you estimate the uncertainty in v? The answer is you use percent uncertainties.

Specifically, suppose that $t = 0.161 \pm 0.014\,\mathrm{s}$ and that $x = 2.0\,\mathrm{cm}$ exactly. (The uncertainty in t is the *standard deviation of the mean.*) You calculate the velocity by using the mean t:

$$v = \frac{x}{t} = \frac{2.0\,\mathrm{cm}}{0.161\,\mathrm{s}} = 12.422\,\mathrm{cm/s}\ .$$

The uncertainty in the velocity is given by σ_v, and can be found from the relation

$$\frac{\sigma_v}{v} = \frac{\sigma_t}{t}. \tag{B.1}$$

The last equation can be solved for σ_v as follows:

$$\sigma_v = v\frac{\sigma_t}{t} = 12.422 \times \frac{0.014}{0.161} = 12.422 \times 0.08696 = 1.1\,\mathrm{cm/s}\;.$$

You would then report your result as

$$v = 12.4 \pm 1.1\,\mathrm{cm/s}\;,$$

where the uncertainty has been rounded to two significant digits, and the value for v has been rounded off to match. Any additional digits in v would be meaningless.

If the value for x also had uncertainty, then Eq. B.1 would be replaced by

$$\frac{\sigma_v}{v} = \sqrt{\left(\frac{\sigma_t}{t}\right)^2 + \left(\frac{\sigma_x}{x}\right)^2}\;.$$

You would solve it the same way, however, by multiplying v over to the right hand side and plugging in your numbers.