Physics 151

Chapter 13 Notes

Gravitational Potential Energy

13 Gravitation

13.3 Gravitational Potential Energy

We have introduced a new force, the gravitational force. Is there a corresponding potential energy? Recall that you could calculate a change in potential energy by considering the work done moving between two positions. If that work is independent of path, you can define a change in potential energy by

$$W = -(U_f - U_i) = -\Delta U$$

Consider two masses, M_1 and M_2. Assume that mass M_1 is at the origin, and mass M_2 is located at $\vec{r}$, a distance r away. The force on M_2 is attractive and points back towards M_1.

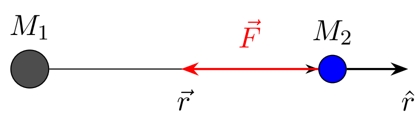


Figure 1: Gravitational force between two masses.

$$\vec{F} = -\frac{GM_1M_2}{r^2}\hat{r} \tag{1}$$

The textbook goes through the details, but the result is fairly simple. The corresponding potential energy (in terms of distance r) is

$$U = -\frac{GM_1M_2}{r} \tag{2}$$

A sketch of U *vs.* r is shown below. Note the we have chosen a standard reference point: When $r \to \infty$, $U \to 0$. This is an arbitrary, but sensible, choice. When two masses are infinitely far apart, it makes sense to say their shared gravitational potential energy is zero.

Note that the signs here are all consistent with what we saw in Ch. 7, where $F = -\frac{dU}{dr}$.

$$\begin{aligned} U &= -\frac{GM_1M_2}{r} \\ F &= -\frac{dU}{dr} = -\frac{d\left(\frac{-GM_1M_2}{r}\right)}{dr} \\ F &= -\frac{GM_1M_2}{r^2} \end{aligned}$$

Thus the potential energy is negative, and the force on M_2 is attractive, that is, to the left.

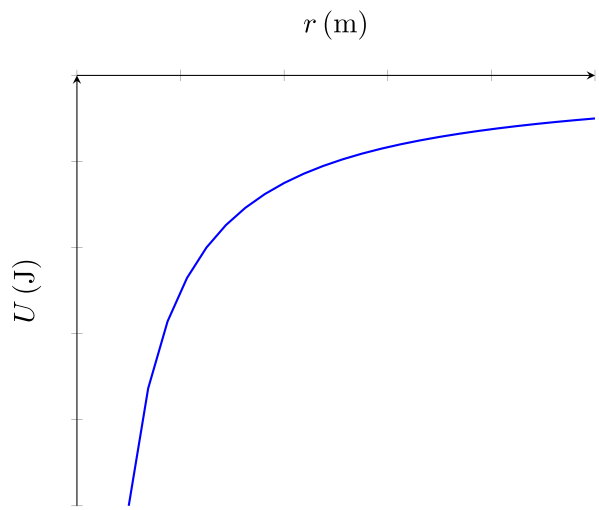


Figure 2: Gravitational potential energy for two masses separated by a distance r.

What about mgy?

In Ch. 7, we wrote the gravitational potential energy near the surface of the Earth as $U_g = mgy$. How does that compare to Eq. 2? If we look at Fig. 2, we see that the potential energy does increase (become less negative) as the distance r increases. One way to see this is to look at the expression relating force and potential, and note that the change in r is typically small. If we consider take a mass m starting on the surface of the Earch at a distance $r_i = R_E$, and go up to a final height y a few meters above the surface of the Earth, then $r_f = R_E + y$ only represents a small change relative to the radius of the Earth.

$$F = -\frac{dU}{dr} \approx -\frac{\Delta U}{\Delta r} \implies \Delta U \approx -F(\Delta r)$$

Then plugging in the expression for the force, note that $\Delta r = y$. For the r in the denominator of the gravitational force, since $y \ll R_E$, it doesn't really matter if we use R_E or $R_E + y$, so just use R_E.

$$\Delta U \approx -\frac{-GM_E m}{R_E^2}(\Delta r) = mgy$$

since $\frac{GM_E}{R_E^2}$ is just the gravitational acceleration at the surface of the Earth, g.

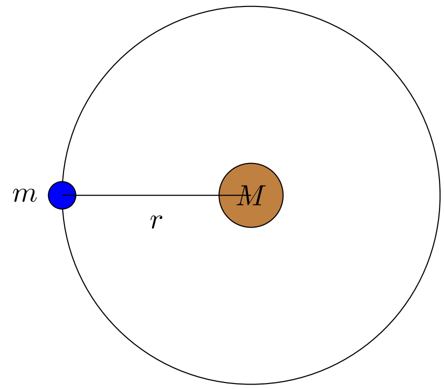
In short, as long as the change in height y is small compared to the radius of the Earth, both $U = -\frac{GM_E m}{r}$ and $U = mgy$ give the same result for ΔU.

13.3.1 Potential Energy in Orbits

Consider an object of mass m in a circular orbit of radius r around a planet of mass M. The orbital velocity is v.

The total energy of mass m is the sum of the kinetic and potential energies:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$



Next, recall that the speed can be found by applying Newton's second law. Rearranging to solve for the kinetic energy, we get the following. (Recall that U is negative, so $-U$ is positive.

$$\begin{aligned}
F &= ma \\
\frac{GMm}{r^2} &= m\frac{v^2}{r} \\
\frac{GMm}{r} &= mv^2 \\
\frac{1}{2}\frac{GMm}{r} &= \frac{1}{2}mv^2 \\
\frac{1}{2}\frac{GMm}{r} &= K \\
-\frac{1}{2}U &= K \\
E = K + U = -\frac{1}{2}U + U &= \frac{1}{2}U = -\frac{GMm}{2r}
\end{aligned}$$

The final result is that the total energy for a circular orbit is negative.

13.3.2 Escape Velocity

Consider an object of mass m on the surface of a planet of mass M and radius R, and launched vertically up with an initial speed v_i. (Ignore the planet's rotation.) What is the initial speed required for the mass m to just barely escape to infinity? This is known as the escape velocity v_e.

$$
\begin{aligned}
E_i &= E_f \\
K_i + U_i &= K_f + U_f \\
\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{R}\right) &= 0 + 0 \\
\frac{1}{2}v_e^2 &= \frac{GM}{R} \\
v_e &= \sqrt{\frac{2GM}{R}}
\end{aligned}
$$

The final kinetic energy is 0 because the mass *just barely* escapes to infinity. It does not have any extra kinetic energy left over. The final potential energy is 0 because the mass m escapes to infinity, so the distance $r \to \infty$.

For Earth, the escape velocity is

$$
\begin{aligned}
&v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2(6.67 \times 10^{-11}\,\mathrm{Nm^2/kg^2})(5.97 \times 10^{24}\,\mathrm{m})}{6.37 \times 10^{6}\,\mathrm{m}}} \\
&v_e = 11\,200\,\mathrm{m/s} \approx 25\,000\,\mathrm{mi/h}
\end{aligned}
$$

13.3.3 Final notes on energy

Note that for the circular orbit, the total energy $E = -\frac{GMm}{2r}$ is negative, while for the partle that just barely escapes, the total energy is zero. More generally, we can classify a particle's motion in the following ways:

State	**Total Energy**
Bound	$E < 0$
Neutral	$E = 0$
Unbound	$E > 0$