Physics 238

The Torsional Oscillator
Part 3: Mechanical Resonance
Report: **Due Friday, March 28, 2025**

Safety

This experiment involves permanent magnets with strong, but localized, magnetic fields. Just outside the wooden frame of the apparatus, the field is approximately 5 mT (*i.e.* 50 Gauss). This may be large enough to interfere with some sensitive medical devices. If you have any concerns, please consult your lab instructor.

1 INTRODUCTION

In previous experiments, you studied the torsional oscillator apparatus as a damped oscillator, and developed tools to measure and characterize the angular position as a function of time. In this experiment, you will add periodic driving to the system, observe how the amplitude and phase of the response vary with driving frequency, and observe resonance.

2 THEORY

2.1 Damped, Driven Oscillations

As before, we assume that the torsional oscillator has a restoring torque τ proportional to the angular displacement $\theta = (\theta_{raw} - \theta_R)$:

$$\tau = -\kappa\theta\,,$$

and that there is a damping proportional to the angular velocity $\dot{\theta}$. In this experiment, we will assume that oscillator is also subject to an oscillating torque, given by

$$\tau_d(t) = \tau_0 \sin(2\pi f_d t) \tag{1}$$

where τ_0 is the amplitude of the driving, and f_d (in Hz) is the frequency.[1] The angular driving frequency is given by $\omega_d = 2\pi f_d$. Although most theory is written in terms of angular frequencies, the function generator used to supply the driving uses Hertz, so that will be more convenient for this lab.

Putting everything together, the differential equation for a damped, driven torsional oscillator of rotational inertia I is

$$\ddot{\theta} = -\omega_0^2\theta - \gamma\dot{\theta} + \alpha_0 \sin(2\pi f_d t) \tag{2}$$

where $\omega_0 = 2\pi f_0 = \sqrt{\frac{\kappa}{I}}$ is the natural (undamped) frequency of the oscillator, $\alpha_0 = \frac{\tau_0}{I}$ is the amplitude of the driving, with units of rad/s^2, and γ is a term proportional to the drag, with units of rad/s.

[1] We could use either sine or cosine, but in this experiment, it will prove convenient to trigger the data collection as the voltage increases through 0 V, so the sine representation will be more natural.

The resulting motion will have an initial transient, but will eventually settle down into a steady state oscillation at the same frequency as the driving frequency, given by

$$\theta(t) = \theta_0 \sin(2\pi f_d t - \phi) \tag{3}$$

where the amplitude θ_0 and phase ϕ of the motion are given by

$$\theta_0 = \frac{\alpha_0}{\sqrt{\left(\omega_0^2 - \omega_d^2\right)^2 + (\gamma\omega_d)^2}} \tag{4}$$

$$\tan\phi = \frac{\gamma\omega_d}{\omega_0^2 - \omega_d^2} \tag{5}$$

Experimentally, it will prove useful to rewrite Eqs. 4 and 5 in terms of the linear frequencies $f = \omega/2\pi$ and the quality factor Q, given by

$$Q = \frac{\omega_0}{\gamma}\,. \tag{6}$$

2.2 Preliminary Calculations

There is a lot of data to take for this lab, so it will be very useful to plan ahead. To get a reasonable plot and fit for Eq. 4, you will need to take a fair amount of data near the resonance peak (in the neighborhood of f_0) and fewer data points further away.

Specifically, assume that $Q = 20$ and that ω_0 is the value you obtained in the previous experiment. Rewrite Eq. 4 in terms of Q and f_0, and plot the predicted ratio $\frac{\theta_0}{\alpha_0}$ *vs.* f_d. Based on that graph, propose a set of approximately 20 frequencies to be measured. You will likely want to adjust them slightly once you have re-measured Q and f_0 for your particular apparatus, but these values should be a good guide.

3 DATA ACQUISITION

3.1 Initial Setup

Use four of the brass quadrants to give the system a reasonably large moment of inertia. Use Channel 1 of the `LabQuest Mini` along with `LoggerPro` to measure the output voltage corresponding to the angular displacement. Let the oscillator come to equilibrium, and adjust the "Zero Adjust" so that the output voltage is close to zero.

Use the following initial settings on `LoggerPro`: Under `Experiment` → `Data Collection`, set the `Experiment Length` to 30 seconds, and the `Sampling Rate` to 500 samples/second. Feel free to adjust these values during the course of the experiment, but these are good for a start.

3.1.1 Initial Estimate of Q

The first step is to adjust the magnets to get a reasonable amount of damping. For this experiment, a value of Q around 20 works well. (This value is not critical; any value in the range $15 \le Q \le 35$ should work fine.)

Move the magnetic brakes near to the rotor. The exact position is not critical. Adjusting the magnets until the leading edge is just over the outer edge of the copper rotor seems to work fine. Be sure the magnets are not touching the rotor.

Connect the DC power supply to the gray terminals marked "Coil Drive." Gradually turn on the current to about 1.0 A and let the rotor settle down. Make sure it is not wobbling.

Set `LoggerPro` to use *triggering* to start data acquisition. (This is available under the `Experiment` → `Data Collection` menu.) Select the tab for `Triggering`, and set it so that data collection will start as the voltage on Ch. 1 increases through 0 V. This will later be convenient for modeling the motion as a sine wave.

Press the `Start` button in `LoggerPro`, and then toggle the `Output` On/Off switch on the power supply to turn off the drive current so that the rotor starts oscillating. Use `LoggerPro` to record the position as a function of time for 30 seconds.

Use `LoggerPro` or *Mathematica* to fit the damped oscillator equation to your data:

$$\theta(t) = \theta_0 e^{-\gamma t/2} \cos(\omega t + \phi) + \theta_{\text{off}} \tag{7}$$

where θ_{off} accounts for any residual zero offset. (Note that to estimate Q, it is not necessary to convert the output voltages to angles.)

Use that fit to estimate the values for f_0 and Q. Technically, the frequency you find in this situation is ω_v, the damped frequency, but recall that the difference between ω_v and ω_0 was very small and difficult to measure reliably with this apparatus, so you may simply assume that the frequency you obtain in Eq. 7 is ω_0.

Make any adjustments to the magnets or apparatus until you are satisfied that you have a reasonable value for Q. Thereafter, be sure not to change the positions of the magnets for the rest of the experiment. You can still safely adjust the "Zero Adjust" knob.

When you are satisfied, *save your data* by exporting it as a "`.csv`" file. You will need it for the analysis below.

3.1.2 Adding Driving

Next, you will drive the system with an oscillating voltage $v_{\text{in}}(t)$, and measure the angular position response $v_{\text{out}}(t)$.

Move the "Angular Position" output to Ch. 2 on the LabQuest Mini. This will be the output voltage v_{out}.

Disconnect the DC power supply. Connect the 50 Ω output of the function generator to the bottom gray and black "Coil Drive" terminals, as in Fig. 1. Note that both the Torsional Oscillator and the function generator have a terminal marked as ground (⏚). These terminals must be connected together. On the BNC adapter, the ground side has a small extra nib (sometimes with a tiny embossed "GND" label) indicating which side should be connected to ground.

Add a jumper cable between the red and top gray "Coil Drive" terminals. Make sure the toggle switch is to the left (for Coil Drive). This configuration ensures that the coil drive current passes through both coils, then through a 1 Ω resistor and finally back to

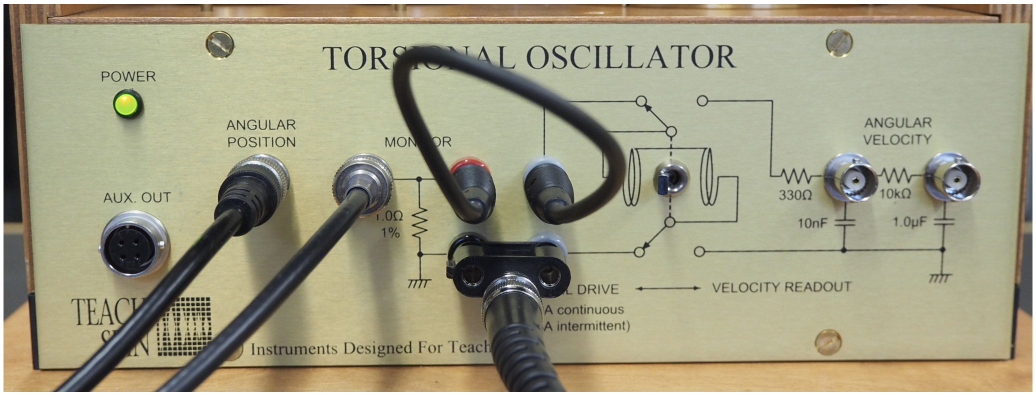

Figure 1: Adding driving for the torsional oscillator.

the function generator. Meanwhile, you can monitor the voltage across that 1 Ω resistor as a way to monitor the driving current (and hence the driving torque). Connect the BNC terminal labeled "Monitor" to Ch. 1 of the LabQuest Mini.

3.1.3 Monitoring Driving and Angular Position

At this point, `LoggerPro` is monitoring the driving voltage v_{in} on Ch. 1 and the response voltage v_{out} on Ch. 2. It is possible to convert each of these voltages to the underlying physical quantities (driving torque and angular displacement), but as long as the resulting angles stay small (under 1 rad, or $|v_{\text{out}}| < 2\,\text{V}$), we can simply assume those conversions are linear, and just use the raw voltages in all future analysis.

Turn on the function generator. Set the "Waveform" to "Sine" waves. Set the frequency to 0.1 Hz (or 100 mHz). You can type the number directly into the keypad and then select the function button next to the appropriate frequency range. Alternatively, you can adjust the frequency with the scroll wheel. You change which digit is active with the left and right arrow keys just below the scroll wheel. Set the peak-to-peak voltage to 8 V. Press the "Output" key to enable the output. You should now observe the oscillator begin to slowly move.

LoggerPro Configuration Use the following initial settings on `LoggerPro`: Under `Experiment` → `Data Collection`, set the `Experiment Length` to 20 s, and the `Sampling Rate` to 500 samples/second. Feel free to adjust these values during the course of the experiment, but these are good for a start. Set `LoggerPro`'s triggering to start collecting data when Ch. 1 is increasing across 0 V.

Collect data. At this low frequency, the driving and the response should be in phase. However, depending on the orientation of the coils that provide the torque, you may see the voltages go in opposite directions. If so, it will prove useful to switch the sign of Ch. 2 now rather than later in your analysis. Specifically, use `Data` → `New Calculated Column` to create a new column. Give it a convenient name, such as "Angular Displacement", and a Short Name, such as "angle". The units are still volts (though that ultimately won't matter in this experiment). In the "Expression" box, type "-1*" and then press the "Variables (Columns)" button to select Ch. 2. Finish by clicking "Done". On your graph, click on the

Y-axis label, select "More ..." and check the boxes for both Ch. 1 and your new column.

Try again to collect new data. Now you should see both the input and output voltages move in sync.

3.1.4 Quick Survey of Phenomena

It is worthwhile to do a quick survey of frequencies to make sure everything is working well. Try a frequency much larger than your resonance frequency. You should observe a small response approximately 180° out of phase with the driving. Lastly, try a frequency close to your expected resonance frequency. You will probably have to decrease the driving amplitude to something like 4 V in order to keep the amplitude of the output voltage less than 2 V (which means the angular displacement is less than about 1 rad, and the sensor is still in its linear regime).

3.2 Data

Ultimately, you want to acquire enough data to test Eqs. 4 and 5 over a reasonably wide range of frequencies, with particular emphasis on frequencies near f_0. That is, you will need to find the amplitude and phase of v_{in} and v_{out} for each driving frequency f_d. A set of about 20 different well-chosen frequencies ought to be sufficient.

Ignoring calibration constants, the driving angular acceleration is given by the Ch. 1 voltage:

$$v_{\text{in}}(t) = V_{\text{in}} \sin\left(2\pi f_d t - \phi_{\text{in}}\right) + V_{\text{off,in}} \,. \tag{8}$$

Ideally, $\phi_{\text{in}} = 0$, but the data acquisition triggering can be affected by noise, as well as by an offset voltage that is not exactly equal to zero.

Similarly, after an initial transient, the angular displacement in the steady state motion is given by v_{out}:

$$v_{\text{out}}(t) = V_{\text{out}} \sin\left(2\pi f_d t - \phi_{\text{out}}\right) + V_{\text{off,out}} \,. \tag{9}$$

The phase difference ϕ between the two signals is simply

$$\phi = \phi_{\text{out}} - \phi_{\text{in}} \,.$$

For each trial, then, you will need to record the driving frequency f_d, the input and output amplitudes, V_{in} and V_{out}, and the input and output phases, ϕ_{in} and ϕ_{out}. You should also export the data for each trial so that you can later import it into *Mathematica* (or other analysis program).

Hints: Note that you can try to do the fits to Eqs. 8 and 9 directly in `LoggerPro`, but you may find that it has difficultly performing reliable fits, or that it sometimes gives you negative amplitudes or frequencies, and that the phase is also sometimes far off. Even in *Mathematica*, though, the fits can sometimes be difficult, and you may need to give `NonlinearModelFit` significant help to find the best solution. (More details are below in the Analysis section.)

If `LoggerPro` inadvertently starts when the voltage is decreasing (so that the phase ϕ_{in} is close to π instead of 0) try a fresh trial. Ultimately, your work will be easier if all the initial phases are about the same, close to zero.

Make sure amplitude of the output voltage is always $< 1.5\,\mathrm{V}$. If the response is too large, simply decrease the amplitude of the driving.

Record the actual amplitude and frequency from the function generator.

Export your data using sensible file names, including the frequency. You will ultimately have about 20 files, so it is important to keep it all organized.

4 ANALYSIS

4.1 Finding Amplitude and Phase Data

For each frequency, fit the input data to a sine wave of the form of Eq. 8. You will likely find it helpful to give `NonlinearModelFit` the frequency from the function generator as the initial guess for the driving frequency f_d. You should also constrain the fit to give a positive value for the amplitude V_{in} and apply a tight constraint for the frequency f_d. For example, if the function generator was set to frequency `fset`, then you can apply those constraints and initial guesses with a command like the following:

```
NonlinearModelFit[data,
  {V0 * Sin[2 \[Pi] fd t - \[Phi]] + Voff, fset-0.01 <= fd <= fset+0.01, V0 > 0},
  {V0, {fd, fset}, \[Phi], Voff}, t]
```

Note that the constraints are included (inside curly braces) along with the equation to fit. You may also sometimes find it helpful to give an initial guess for the phase ϕ. You will likely have to tweak these settings for different input files.

Similarly, fit the output data to a sine wave of the form of Eq. 9.

Make a table showing f_d, V_{in}, ϕ_{in}, V_{out}, ϕ_{out},

4.1.1 Graphs

In your final report, include a graph showing both the raw data v_{in} and v_{out}, along with the corresponding fits, for three different frequencies: one frequency significantly below f_0, one frequency near f_0, and one frequency significantly above f_0. Discuss the qualitative differences.

4.2 Resonance Curve

Use Eq. 4 to find an equation for the ratio $\dfrac{V_{\text{out}}}{V_{\text{in}}}$ in terms of linear frequencies f_0 and f_d and quality factor Q. Instead of working out all the calibration constants, just include an overall constant C in the numerator.

Fit your revised equation to your data. Make a graph showing your data along with the fit. Give the values for f_0 and Q, along with their uncertainties.

4.3 Phase Shift

Use Eq. 5 to find an equation for the phase difference ϕ in terms of f_0, f_d, and Q.

Fit your revised equation to your data. Make a graph showing your data along with the fit. You may find the two-argument form of *Mathematica*'s `ArcTan[x, y]` function most useful here. Give the values for f_0 and Q, along with their uncertainties.

4.4 Overall Comparison

Compare your results for f_0 and Q from the resonance curve, the phase curve, and the results from §3.1.1 above. Discuss any significant differences.