Phys 238 HW #4: Resonance: FWHM and Mean Lifetime

1. Assignment 1

$$\theta_0^2 = \frac{\alpha_0^2}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2}$$

what value for ω_d maximizes θ_0^2 ?

Set $\frac{d\,\theta_0^2}{d\omega_d} = 0$

$$-1\frac{\alpha_0^2}{\left[(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2\right]^2} \cdot \left(2(\omega_0^2-\omega_d^2)(-2\omega_d) + 2\gamma^2\omega_d\right) = 0$$

$$2(\omega_0^2 - \omega_d^2)(2\omega_d) = 2\gamma^2\omega_d$$

$$\omega_0^2 - \omega_d^2 = \frac{\gamma^2}{2}$$

$$\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} = \omega_0\sqrt{1 - \frac{\gamma^2}{2\omega_0^2}}$$

peak is at $\boxed{\omega_p = \omega_0\sqrt{1-\frac{\gamma^2}{2\omega_0^2}}} = \omega_0\sqrt{1-\frac{1}{2Q^2}}$

Height of peak is

$$\theta_p^2 = \frac{\alpha_0^2}{(\omega_0^2 - \omega_p^2)^2 + (\gamma\omega_p)^2} = \frac{\alpha_0^2}{\left(\frac{\gamma^2}{2}\right)^2 + (\gamma\omega_p)^2}$$

$$\theta_p^2 = \frac{\alpha_0^2}{\frac{\gamma^4}{4} + \gamma^2\omega_p^2} = \frac{\alpha_0^2}{\frac{\gamma^4}{4} + \gamma^2\omega_0^2\left(1 - \frac{\gamma^2}{2\omega_0^2}\right)}$$

$$\theta_p^2 = \frac{\alpha_0^2}{\frac{\gamma^4}{4} + \gamma^2\omega_0^2 - \frac{\gamma^4}{2}} = \frac{\alpha_0^2}{\gamma^2\omega_0^2 - \underbrace{\frac{\gamma^4}{4}}_{\text{Drop this term}}}$$

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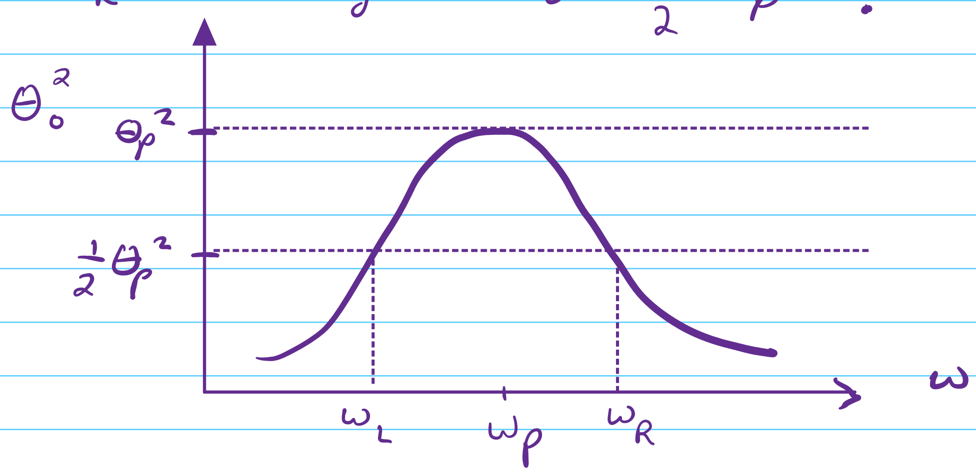
Note:
$$\gamma^2\omega_0^2 - \frac{\gamma^4}{4} = \omega_0^4\left(\frac{\gamma^2}{\omega_0^2} - \underbrace{\frac{\gamma^4}{4\omega_0^4}}_{\text{Drop this term.}}\right)$$

$$\boxed{\theta_p^2 \simeq \frac{\alpha_0^2}{\gamma^2\omega_0^2}}$$

#2 What are the two frequencies (call them ω_L and ω_R) that give $\theta_0^2 = \frac{1}{2}\theta_p^2$?



$$\theta_0^2 = \frac{\alpha_0^2}{(\omega_0^2-\omega_d^2)^2 + (\gamma\omega_d)^2} = \frac{1}{2}\frac{\alpha_0^2}{\gamma^2\omega_0^2}$$

Cross multiply

$$2\gamma^2\omega_0^2 = (\omega_0^2-\omega_d^2)^2 + \gamma^2\omega_d^2$$
$$2\gamma^2\omega_0^2 = \omega_0^4 - 2\omega_0^2\omega_d^2 + \omega_d^4 + \gamma^2\omega_d^2$$

$$\omega_d^4 - (2\omega_0^2 - \gamma^2)\,\omega_d^2 + (\omega_0^4 - 2\gamma^2\omega_0^2) = 0$$

This is a quadratic in ω_d^2:
$$\omega_d^4 - b\,\omega_d^2 + c = 0$$

where "b" $= 2\omega_0^2 - \gamma^2$ and "c" $= \omega_0^4 - 2\gamma^2\omega_0^2$

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$$\omega_d^2 = \frac{b \pm \sqrt{b^2 - 4c}}{2}$$

$$2\omega_d^2 = (2\omega_0^2 - \gamma^2) \pm \sqrt{(2\omega_0^2 - \gamma^2)^2 - 4(\omega_0^4 - 2\gamma^2\omega_0^2)}$$

$$2\omega_d^2 = (2\omega_0^2 - \gamma^2) \pm \sqrt{4\omega_0^4 - 4\omega_0^2\gamma^2 + \gamma^4 - 4\omega_0^4 + 8\gamma^2\omega_0^2}$$

$$2\omega_d^2 = (2\omega_0^2 - \gamma^2) \pm \sqrt{4\omega_0^2\gamma^2 + \gamma^4}$$

This term will be small. Here is a careful approach:

$$2\omega_d^2 = (2\omega_0^2 - \gamma^2) \pm 2\omega_0\gamma\sqrt{1 + \frac{\gamma^4}{4\omega_0^2\gamma^2}}$$

$$= (2\omega_0^2 - \gamma^2) \pm 2\omega_0\gamma\sqrt{1 + \frac{\gamma^2}{4\omega_0^2}}$$

Expand for $\gamma \ll \omega_0$.

$$= (2\omega_0^2 - \gamma^2) \pm 2\omega_0\gamma\left(1 + \frac{1}{2}\frac{\gamma^2}{4\omega_0^2}\right)$$

$$= 2\omega_0^2\left[1 - \underbrace{\frac{\gamma^2}{2\omega_0^2}}_{\text{Drop}} \pm \frac{\gamma}{\omega_0} \pm \underbrace{\frac{1}{8}\frac{\gamma^3}{\omega_0^3}}_{\text{Drop}}\right] = 2\omega_0^2\left[1 \pm \frac{\gamma}{\omega_0}\right]$$

$$\omega_d = \omega_0\sqrt{1 \pm \frac{\gamma}{\omega_0}} = \omega_0\left(1 \pm \frac{1}{2}\frac{\gamma}{\omega_0}\right) = \omega_0 \pm \frac{\gamma}{2}$$

Low frequency $\omega_L = \omega_0 - \dfrac{\gamma}{2}$

high frequency $\omega_R = \omega_0 + \dfrac{\gamma}{2}$

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#3 $\boxed{FWHM = \omega_R - \omega_L = \gamma = \frac{1}{\tau}}$

#4 Torsional Oscillator:

I got from damped oscillation:

$\omega_0 = 4.788$ rad/s

$\gamma = 0.1312$ /s

$Q = \frac{\omega_0}{\gamma} = 36.5$

$\tau = \frac{1}{\gamma} = 7.623$ s

From the resonance curve: (Looking for $\theta_0 = \frac{1}{\sqrt{2}}\,\theta_{peak}$)

$f_L = 0.747$ Hz

$f_R = 0.770$ Hz

$\Delta f = 0.023$ Hz

$\Delta\omega = 2\pi\,\Delta f = 0.145$ rad/s $=$ FWHM

$\frac{1}{FWHM} = \frac{1}{0.145 \text{ rad/s}} = 6.92$ s

These two values are close, but not exactly the same.

#5 LRC Resonance

From the damped oscillation:

$\gamma = 12215 \pm 25$ /s

$\tau = 8.19 \times 10^{-5}$ s

From resonance curve:

$f_L = 7770$ Hz $\qquad f_R = 9730$ Hz

$\Delta\omega = 2\pi\Delta f = FWHM = 12314$ rad/s

$\frac{1}{FWHM} = 8.12 \times 10^{-5}$ s. Again, these are close.