Resonance

Consider the damped torsional oscillator with moment of inertia I, restoring torsional constant K, and damping γ given by

$$\ddot{\theta} = -\omega_0^2 \theta - \gamma \dot{\theta}$$

where $\omega_0 \equiv \sqrt{\frac{K}{I}}$.

We saw that the solution for the underdamped case could be written

$$\theta = \theta_0 e^{-\gamma t/2} \sin(\omega_v t + \phi)$$

where

$\omega_v = \omega_0 \sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}$, though the difference between ω_v and ω_0 was too small to reliably measure.

Lastly, we defined a quality factor "Q" by

$$Q = \frac{\omega_0}{\gamma}.$$

Adding Driving

Next, suppose we add a periodic driving torque

$\tau_0 \sin(\omega_d t)$ where

ω_d is the driving frequency, which can be adjusted. The differential equation becomes

$$\ddot{\theta} = -\omega_0^2 \theta - \gamma \dot{\theta} + \alpha_0 \sin(\omega_d t)$$

where $\alpha_0 = \tau_0 / I$.

Resonance

Simulation: After an initial transient, we observe that the system settles down into sinusoidal oscillations at the same frequency as the driving.

Assume steady state oscillation is

$$\theta(t) = \theta_0 \sin(\omega_d t - \phi)$$

Plug it in and try it - solve for θ_0 and ϕ.

This will be easier with the complex exponential

Driving is of the form $\alpha_0 e^{i\omega_d t}$.

Assume $\theta(t) = \theta_0 e^{i(\omega_d t - \phi)}$

$$\dot{\theta}(t) = i\theta_0 \omega_d e^{i(\omega_d t - \phi)}$$

$$\ddot{\theta}(t) = -\omega_d^2 \theta_0 e^{i(\omega_d t - \phi)}$$

Plug in. Cancel the common factor of $e^{i(\omega_d t - \phi)}$.

$$-\omega_d^2 \theta_0 = -\omega_0^2 \theta_0 - i\gamma\theta_0\omega_d + \alpha_0 e^{i\phi}$$

$$-\omega_d^2 \theta_0 = -\omega_0^2 \theta_0 - i\gamma\theta_0\omega_d + \alpha\cos\phi + i\alpha\sin\phi$$

This is 2 equations: Real part and Imaginary part.

Real:

$$-\omega_d^2 \theta_0 = -\omega_0^2\theta_0 + \alpha\cos\phi$$

$\Rightarrow$

$$\alpha_0 \cos\phi = (\omega_0^2 - \omega_d^2)\,\theta_0$$

Imaginary:

$$\alpha_0 \sin\phi = \gamma\theta_0\omega_d$$

Resonance

Square and add.

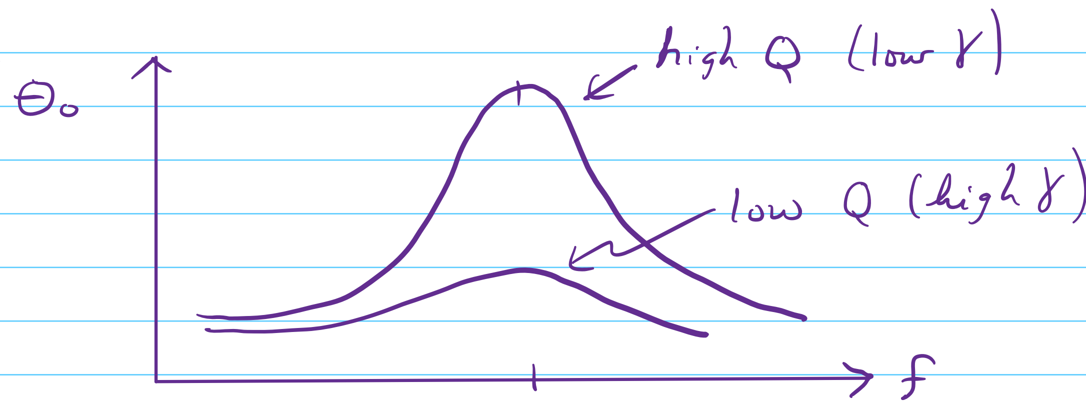
$$\alpha_0^2(\cos^2\phi + \sin^2\phi) = (\omega_0^2 - \omega_d^2)^2\theta_0^2 + \gamma^2\omega_d^2\theta_0^2$$

$$\boxed{\theta_0 = \frac{\alpha_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2}}}$$

Take ratio

$$\frac{\alpha \sin\phi}{\alpha \cos\phi} = \boxed{\frac{\gamma\omega_d}{(\omega_0^2 - \omega_d^2)} = \tan\phi}$$

Plots:



Peak is near $\omega_d = \omega_0$. Or, it will be easier to use linear frequencies $f_d = \frac{\omega_d}{2\pi}$

$$\theta_0 = \frac{\alpha_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2}}$$

Use $Q = \frac{\omega_0}{\gamma} \Rightarrow$

$$\gamma = \frac{\omega_0}{Q}$$

$$\theta_0 = \frac{\alpha_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \left(\frac{\omega_0\omega_d}{Q}\right)^2}}$$

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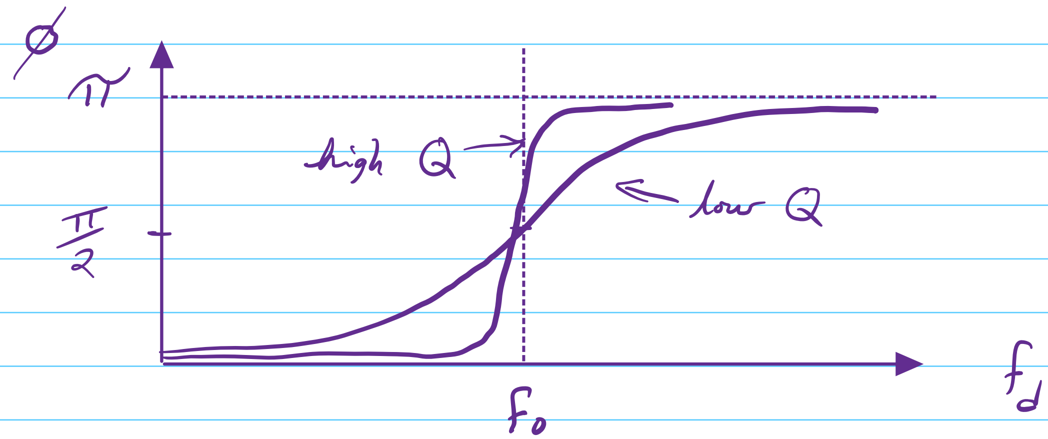
$$\theta_0 = \frac{\alpha_0 / 4\pi^2}{\sqrt{(f_0^2 - f_d^2) + \left(\frac{f_0 f_d}{Q}\right)^2}}$$



Note: ArcTan[y/x] returns $-\pi/2$ to $\pi/2$
ArcTan[x, y] returns $-\pi$ to π
since it has full quadrant information.

$$\tan\phi = \frac{\gamma \omega_d}{\omega_0^2 - \omega_d^2}$$

our numerator is always positive, but the denominator changes sign, so our ϕ is in the range 0 to π
(i.e. our "y" value is always positive.)