Phys 335 Problem 3.5

Problem 3.5. Starting with the result of Problem 2.17, find a formula for the temperature of an Einstein solid in the limit $q \ll N$. Solve for the energy as a function of temperature to obtain $U = N\epsilon e^{-\epsilon/kT}$ (where ϵ is the size of an energy unit).

Start (problem 2.17) , $q \ll N$

$$\Omega(N,q) \approx \left(\frac{eN}{q}\right)^q$$

$$S = k \ln \Omega = kq \ln\left(\frac{eN}{q}\right) = kq\left[1 + \ln\left(\frac{N}{q}\right)\right]$$

$$U = q\epsilon \Rightarrow q = U/\epsilon$$

$$S = \frac{kU}{\epsilon}\left[1 + \ln\left(\frac{N\epsilon}{U}\right)\right] = \frac{kU}{\epsilon}\left[1 + \ln(N\epsilon) - \ln U\right]$$

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k}{\epsilon}\left[1 + \ln N\epsilon - \ln U\right] + \frac{kU}{\epsilon}\left[-\frac{1}{U}\right]$$

$$\frac{1}{T} = \frac{k}{\epsilon} \ln \frac{N\epsilon}{U}$$

$$\frac{\epsilon}{kT} = \ln \frac{N\epsilon}{U}$$

$$e^{\epsilon/kT} = \frac{N\epsilon}{U} \Rightarrow U = N\epsilon e^{-\epsilon/kT}.$$