Physics 131
Chapter 9 Notes—Part 1
Rotation of Rigid Bodies

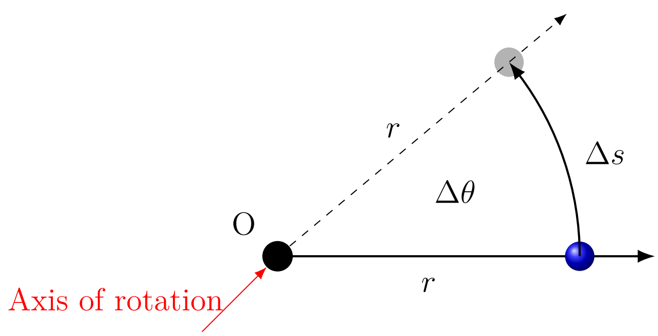
9 Rotation of Rigid Bodies

Introduction

So far, we have developed the tools to describe translational motion (*i.e.* x, y, and z motion) of particles. In this chapter, we will add the possibility of rotation as well. For most of the chapter, we will stick to the case of rotation about a fixed axis, and consider how to describe motion involving *angular* position, velocity, and acceleration.

9.1 Angular Velocity and Acceleration

The text defines angular position, angular velocity, and angular acceleration carefully; refer there for details. The main results are as follows. Consider a point on the rim of a circle of radius r traveling around in a circle. If the point moves an arc length distance Δs, then the angle $\Delta\theta$ is *defined* by $\Delta\theta = \dfrac{\Delta s}{r}$.



Name	Symbol	Definition	Units
Radius	r		meters
Arc Length	s		meters
Angular Position	θ	$\dfrac{s}{r}$	radians
Angular Velocity	ω	$\dfrac{d\theta}{dt}$	radians/s
Angular Acceleration	α	$\dfrac{d\omega}{dt}$	radians/s^2

It is important to note that we will use *radians* for angular displacements, not degrees. The relation is:

$$\theta_{\text{full circle}} = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians}$$

Revolutions, radians, and degrees are related by:

$$1\,\text{rev} = 2\pi\,\text{radians} = 360\,\text{degree}$$

For a rigid body, such as a bicycle wheel, different points have different velocities (both the magnitude and direction can differ), but all rotate together with the same angular velocity ω and acceleration α.

Rotation with Constant Angular *Velocity*

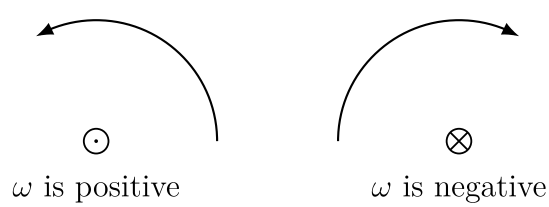
In Chapter 3, we considered uniform circular motion. In the language of Ch. 9, this would be motion with $\omega =$ constant, and, hence, $\alpha = 0$. Since ω is in radians/second, and since it takes 2π radians to complete one revolution, we can also define the related quantities $T = \frac{2\pi}{\omega}$ is the period, or time it takes to rotate through and angle of 2π. (Thus $2\pi = \omega T$.) Similarly, $f = \frac{1}{T} = \frac{\omega}{2\pi}$ is the frequency, *i.e.* how many revolutions the object makes per second. These ideas can be summarized:

$$\omega = 2\pi f = \frac{2\pi}{T}$$

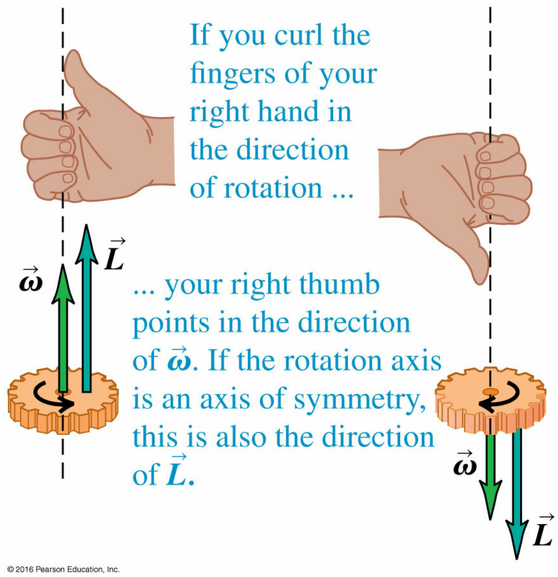
It can be confusing, sometimes, to have three different ways to talk about the same thing. It is important to read problems carefully, and to be mindful of factors of 2π.

Sign Convention

By convention, we usually call the counter-clockwise direction positive, and the clockwise direction negative.



It is also sometimes useful to associate the direction with a right hand rule: Take your right hand: Curl your fingers in the direction of motion. If your thumb points up, call it positive. (The symbol is $\odot$ – think of an arrow coming up out of the page.) If your thumb points down, call it negative. (The symbol is $\otimes$ – think of an arrow heading into the page.)



9.2 Rotation with Constant Angular Acceleration

In Ch. 2, we found the equations governing motion with constant linear acceleration. Because of the analogy between angular and linear motion, very similar equations hold here as well. It is helpful to make the analogy between them:

$$\begin{aligned} \theta &\Leftrightarrow x \\ \omega &\Leftrightarrow v \\ \alpha &\Leftrightarrow a \end{aligned}$$

$$\begin{aligned} \alpha &= \text{constant} \\ \omega &= \omega_0 + \alpha t \\ (\theta - \theta_0) &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega^2 &= \omega_0^2 + 2\alpha\,(\theta - \theta_0) \end{aligned} \tag{1}$$

Remember that for a rigid body, all points travel the same angular distance $\Delta\theta = (\theta - \theta_0)$, have the same angular velocity ω, and the same angular acceleration α. We will usually take the initial angle θ_0 to be 0.

For examples, see the text, and the posted examples Ch09-centrifuge-1.pdf and Ch09-discus.pdf. Most problems we will encounter will use the set of equations in Eq. 1 in the context of a broader problem.

9.3 Relating Linear and Angular Kinematics

The central idea is that particles further from the axis of rotation have to travel further in order to go around once.

The main results are:

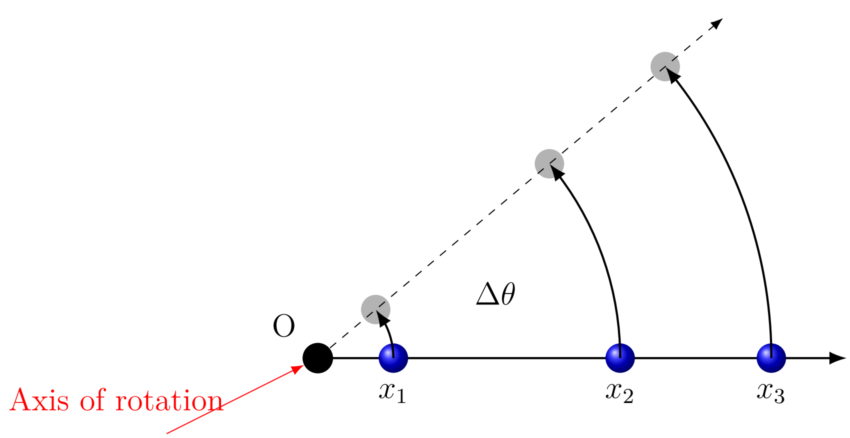
Arc Length	s	$=$	$r\theta$	$r\times$Angular distance
Linear Velocity	v	$=$	$r\omega$	$r\times$Angular velocity
Tangential Acceleration	a_{tan}	$=$	$r\alpha$	$r\times$Angular acceleration
Centripetal Acceleration	a_{rad}	$=$	$\omega^2 r$	

It is important to remember the difference between the the centripetal acceleration $a_{rad} = \omega^2 r$ and the tangential acceleration $a_{tan} = r\alpha$. Any particle traveling in a circle, even at constant speed, has a constantly changing direction for the velocity vector $\vec{v}$, and hence has an acceleration component $a_{rad} = \frac{v^2}{r} = \omega^2 r$ pointing towards the center. In addition, the particle may be speeding up or slowing down, which would require a *tangential* acceleration, $a_{tan} = r\alpha$.

9.4 Energy in Rotational Motion

In a rigid body, all particles have the same angular velocity and acceleration. It is probably easiest to first think about a few masses along a single axis, rotating around in a circle.

Consider a set of masses $\{m_1, m_2, m_3\}$, at positions $\{x_1, x_2, x_3\}$ on a rigid massless rod along the x-axis. Now suppose the whole thing is rotating about the origin O with angular velocity ω. All particles have the same angular velocity ω since they rotate together as a rigid object, but they have different linear velocities $v = r\omega$.



$$
\begin{aligned}
K &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 \\
K &= \frac{1}{2}\left(m_1v_1^2 + m_2v_2^2 + m_3v_3^2\right) \\
K &= \frac{1}{2}\left(m_1(r_1\omega)^2 + m_2(r_2\omega)^2 + m_3(r_3\omega)^2\right) \\
K &= \frac{1}{2}\left(m_1r_1^2 + m_2r_2^2 + m_3r_3^2\right)\omega^2 \\
K &= \frac{1}{2}\left(\sum_{i=1}^{3} m_ir_i^2\right)\omega^2 \\
K &= \frac{1}{2}I\omega^2
\end{aligned}
$$

where in the final line, we have defined the moment of inertia

$$I = \sum_i m_ir_i^2$$

The moment of inertia I depends on the amount of mass as well as its distribution. If all the masses were closer to the axis of rotation, I would be smaller and it would take less energy to make it rotate. If they were further away, I would be larger and it would take more energy to make it rotate.

Moments of inertia for common shapes

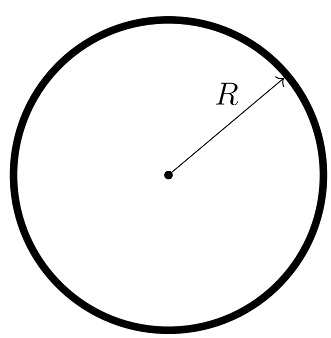
Table 9.2 (included below) shows the moment of inertia for common shapes. It is generally of the form

$$I = cML^2$$

where M is the total mass, L is some measure of the size, and c is a constant that depends on the distribution of mass: c is smaller if the mass is mostly close to the axis, while c is larger if the mass is mostly further away from the axis.

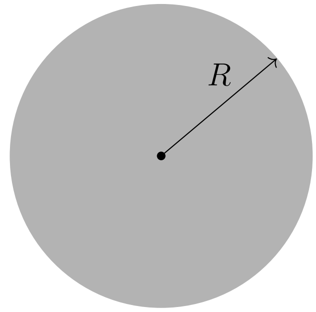
For example, for a hoop, all the mass is a distance R away from the center, so

$$I = MR^2.$$



On the other hand, for a solid disk, the mass is spread out over a range of distances from 0 to R away from the center, so the moment of inertial will be somewhat smaller. We expect $I_{\text{disk}} < I_{\text{hoop}}$. The result is

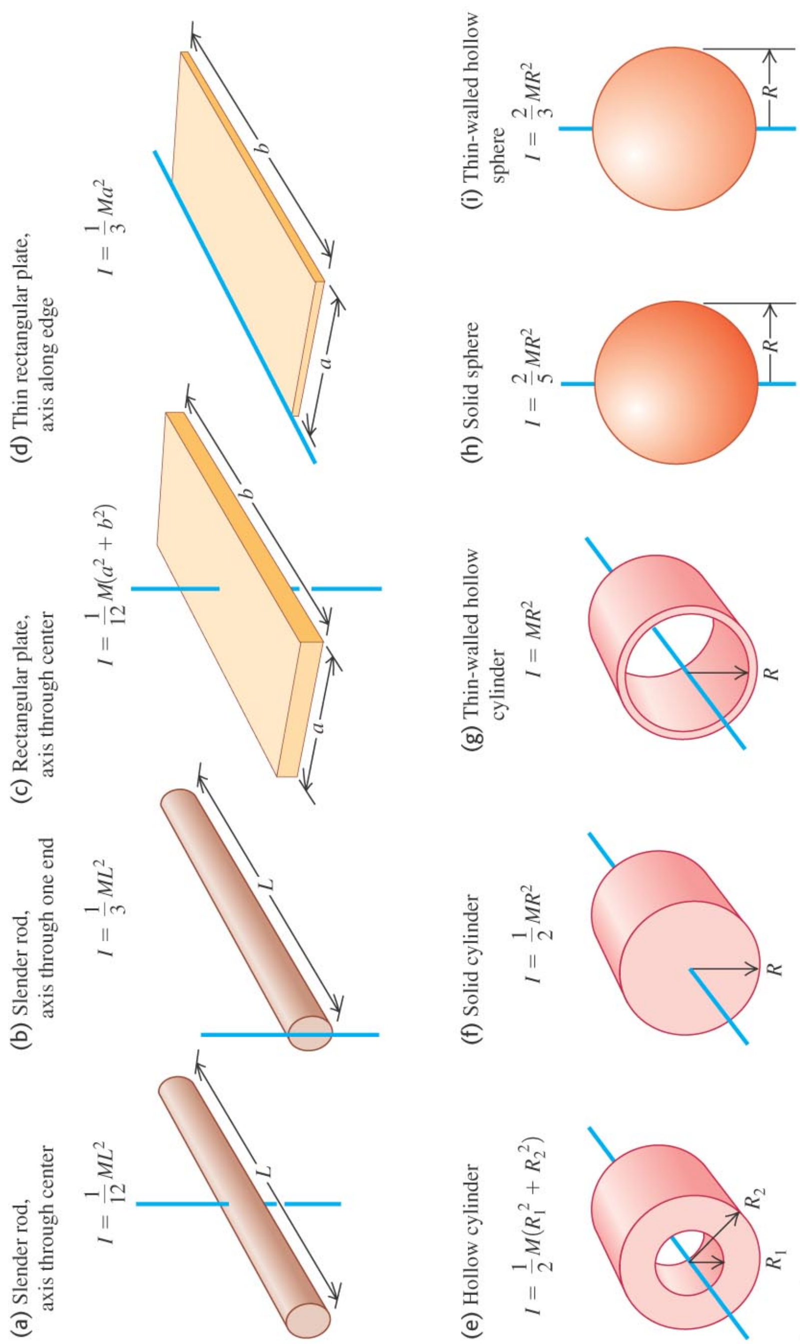
$$I = \frac{1}{2}MR^2.$$



Details on how to calculate I for various shapes are in §9.6.

Examples

Examples 9.6, 9.7, and 9.8 in the text are worth reading. There are also two posted Chapter 9 examples from old tests that include rotational energy. The rotational energy lab is another good application. Additional examples in Chapter 10 will explore problems from the point of view of torque as well as energy.


(a) Slender rod, axis through center
$I = \frac{1}{12}ML^2$
L
(b) Slender rod, axis through one end
$I = \frac{1}{3}ML^2$
L
(c) Rectangular plate, axis through center
$I = \frac{1}{12}M(a^2 + b^2)$
a
b
(d) Thin rectangular plate, axis along edge
$I = \frac{1}{3}Ma^2$
a
b
(e) Hollow cylinder
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
R_1
R_2
(f) Solid cylinder
$I = \frac{1}{2}MR^2$
R
(g) Thin-walled hollow cylinder
$I = MR^2$
R
(h) Solid sphere
$I = \frac{2}{5}MR^2$
R
(i) Thin-walled hollow sphere
$I = \frac{2}{3}MR^2$
R